

STABILITY OF RESTRICTIONS OF LAZARSFELD-MUKAI BUNDLES VIA WALL-CROSSING, AND MERCAT'S CONJECTURE

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ABSTRACT. We use wall-crossing with respect to Bridgeland stability conditions to prove slope-stability of restrictions of locally free sheaves to curves on the K3 surfaces. As a result, we find many new counterexamples to Mercat's conjecture for vector bundles of rank greater than two.

1. INTRODUCTION

Lazarsfeld-Mukai bundles on a K3 surface, and their restriction to curves, have been used for many different applications. Recently, they have been appeared as counterexamples to Mercat's conjecture for vector bundles of rank 3 and 4 which requires slope-stability of restrictions of these bundles, see [FO12, MAO14]. In this paper, we extend these results to any rank greater than 2 by using Bridgeland stability conditions.

Overview. Let $\mathcal{U}_C(n, d)$ be the set of semistable vector bundles of rank n and degree d on a smooth curve C . Then for $E \in \mathcal{U}_C(n, d)$, Clifford index is defined as

$$\text{Cliff}(E) = \mu(E) - \frac{2}{n}h^0(C, E) + 2 \geq 0.$$

The rank n Clifford index of C is defined as

$$(1) \quad \text{Cliff}_n(C) = \min\{\text{Cliff}(E) : E \in \mathcal{U}_C(n, d), d \leq n(g-1), h^0(C, E) \geq 2n\}.$$

Clearly, we have $\text{Cliff}_n(C) \leq \text{Cliff}_1(C)$. However, Mercat conjectured that we have equality for any smooth curve C [Mer02]:

$$(M_n) : \text{Cliff}_n(C) = \text{Cliff}_1(C)$$

Assume X is a smooth complex algebraic K3 surface, and let $C \subset X$ be a smooth curve on the surface. For a globally generated line bundle A on the curve C , the Lazarsfeld-Mukai bundle $E_{C,A}$ is defined via the exact sequence

$$(2) \quad 0 \rightarrow E_{C,A}^\vee \rightarrow H^0(C, A) \otimes \mathcal{O}_X \rightarrow A \rightarrow 0.$$

The bundles $E_{C,A}$ have been appeared, for example, in Lazarsfeld's proof of Brill-Noether-Petri Theorem [Laz86] or in Voisin's proof of Green's canonical syzygy conjecture [Voi05]; see [Apr13] for a survey of applications. In addition, the restriction of Lazarsfeld-Mukai bundle to smooth curves on a K3 surface has led to counterexamples for Mercat's conjecture [FO12, MAO14].

Main result. Let (X, H) be a smooth polarized K3 surface over \mathbb{C} . We say the pair (X, H) satisfies condition $(*)$ if

$$\text{for any curve } C' \subset X, \quad (H^2)|(H.C'). \quad (*)$$

For instance, a polarized K3 surface (X, H) satisfies condition $(*)$ if $\text{Pic}(X) = \mathbb{Z}.H$. Recall that a vector bundle E on X is μ_H -stable if for each proper quotient sheaf E' we have $\mu_H(E) < \mu_H(E')$, where $\mu_H(E) = \frac{c_1(E).H}{\text{rk}(E)}$ is the slope of E .

Theorem 1.1. *Let (X, H) be a smooth polarized K3 surface satisfying condition $(*)$. Let F be a μ_H -stable locally free sheaf on X with Mukai vector*

$$v(F) = (\text{rk}(F), c_1(F), ch_2(F) + \text{rk}(F)),$$

where $\text{rk}(F)$ and $\frac{c_1(F).H}{H^2}$ are coprime and $\text{rk}(F) > 1$. Then the restriction sheaf, $F|_C$ is slope stable for any curve $C \in |H|$ if

$$(3) \quad H^2 + \frac{H^2(\text{rk}(F) - 2)}{(\text{rk}(F) - 1)^2} - 2\text{rk}(F)^2 > \Delta_H(F),$$

where $\Delta_H(F) = \frac{(c_1(F).H)^2}{H^2} - 2\text{rk}(F)(ch_2(F) + \text{rk}(F))$.

Corollary 1.2. *Let the smooth curves $C, C' \in |H|$ have genus g , and let A be a globally generated line bundle on the curve C with $h^0(C, A) = r + 1 \geq 1$ and $\deg_C(A) = d$. Then, the restriction $E_{C,A}|_{C'}$ of the Lazarsfeld-Mukai bundle of A is slope-stable if*

$$(4) \quad 1 + \frac{r^2(r+1)}{r^2(r+1) + (r-1)} d < g.$$

In particular, for $r > 1$ and $g \geq d + 1$, the vector bundle $E_{C,A}|_{C'}$ is stable. Moreover, by Lazarsfeld's Brill-Noether theorem [Laz86] existence of a line bundle on the smooth curve C with $r + 1$ global sections and degree d is equivalent to an upper bound for g :

$$(5) \quad \rho(r, d, g) \geq 0 \quad \Rightarrow \quad g \leq \frac{r+1}{r} d - (r+1).$$

Corollary 1.3. *Assume the pair (X, H) and the smooth curves $C, C' \in |H|$ are as above. Then restriction of Lazarsfeld-Mukai bundle $E_{C,A}|_{C'}$ invalidates Mercat's conjecture (M_{r+1}) if*

(a) $r + 1 = 3$ and

(i) g is odd, $g \geq 9$ and

$$\frac{4}{3}d - 3 < g \leq \frac{3}{2}d - 3,$$

(ii) or g is even and

$$\frac{4}{3}d - 2 \leq g \leq \frac{3}{2}d - 3; \quad \text{or}$$

(b) $r + 1 > 3$ and

$$1 + d \leq g \leq \frac{r+1}{r}d - (r+1).$$

For any smooth curve $C \in |H|$ and given integers $r \geq 2$ and d which satisfy the assumption in the corollary, there exists a line bundle A on the curve C with $h^0(C, A) = r + 1$ and $\deg_C(A) = d$ such that the corresponding sheaf $E_{C,A}|_{C'}$ is a counterexample for the Mercat conjecture (M_{r+1}) . In fact, Corollary 1.3 provides all the possible cases where the restriction $E_{C,A}|_{C'}$ of Lazarsfeld-Mukai bundles invalidate Mercat's conjecture.

Relation to Previous work. It has been proved that (M_2) holds for a general curve and for a smooth curve $C \in |H|$ on a K3 surface X with $\text{Pic}(X) = \mathbb{Z} \cdot H$ [BF15]. However, counterexamples to (M_2) have been found using curves on K3 surfaces of higher Picard rank, see [FO12], [MAO14], and [LN11].

As proven in [FO12], for a K3 surface X with $\text{Pic}(X) = \mathbb{Z} \cdot H$, if A is a line bundle on $C \in |H|$ with $h^0(C, A) = 3$, then the restriction of Lazarsfeld-Mukai bundle $E_{C,A}|_C$ is stable if

$$\deg_C(A) = \lfloor \frac{2g+8}{3} \rfloor \quad \text{and} \quad g = 7, 9 \text{ or } g \geq 11.$$

Also, it invalidates the Mercat conjecture (M_3) if $g = 9$ or $g \geq 11$.

However, Corollary 1.2 and inequality (5) show that for $g \geq 9$ and any value of d (which there exists at least one) satisfying

$$\frac{2}{3}g + 2 \leq d < \frac{13}{12}g - \frac{13}{12},$$

the bundle $E_{C,A}|_C$ is stable. It is also a counterexample to (M_3) under the assumption in Corollary 1.3.

It has been also shown in [MAO14] that for a K3 surface X with $\text{Pic}(X) = \mathbb{Z} \cdot C$ and line bundle A on C with $h^0(C, A) = 4$ whenever

$$d + 2 \leq g \leq \frac{4}{3}d - 4,$$

the bundle $E_{C,A}|_C$ is slope-stable. Corollary 1.2 gives a better lower bound for g .

There are also some other results which use different techniques, such as taking evaluation map on the curve instead of the surface to find counterexample for (M_3) [LMN12], or restricting the bundle $E_{C,A}$ to a curve of higher degree to show existence of a counterexample for (M_n) when $n > 3$ [Sen16]. But, we show that for any smooth curve $C \in |H|$ with genus g , the Mercat's conjecture M_n fails for $4 \leq n < \sqrt{g}$ and M_3 fails where $g = 9$ or $g \geq 11$.

Strategy of the proof. In order to prove Theorem 1.1, we use stability conditions on the bounded derived category of coherent sheaves on X and wall-crossing, see [Bri08, BM14a, BM14b].

The slope-stability of vector bundle F shows that there are stability conditions σ_1 and σ_2 such that F and $F(-H)$ are σ_1 and σ_2 -stable, respectively. Also, if inequality (3) satisfies, there exists a stability condition σ_3 such that F and $F(-H)[1]$ have the same phase. Then,

we show that F and $F(-H)$ remain stable on the paths which connect stability conditions σ_1 and σ_2 to σ_3 . Hence, they both are σ_3 -stable. Now, the distinguished triangle

$$F \rightarrow F|_C \rightarrow F(-H)[1]$$

gives σ_3 -semistability of $F|_C$ for $C \in |H|$. Finally, by changing σ_3 in the right direction, we can reach strict stability of $F|_C$. Then a general argument immediately implies that $F|_C$ is slope-stable.

Slope stability of tangent bundle of \mathbb{P}^n restricted to a surface. In the second part of the paper, we use similar methods, to reprove Camere's result on the stability of the vector bundle M_L , which is defined as follows. Let X be an algebraic K3 surface over \mathbb{C} , which not necessarily satisfies condition $(*)$, and L be a globally generated ample line bundle on X . Assume M_L is the kernel of evaluation map on the global sections of L :

$$(6) \quad 0 \rightarrow M_L \rightarrow H^0(X, L) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} L \rightarrow 0.$$

Theorem 1.4. [Cam12, Theorem 1] *Assume X is a complex algebraic K3 surface and L is a globally generated ample line bundle on X . Then the vector bundle M_L is μ_L -stable.*

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2. REVIEW: GEOMETRIC STABILITY CONDITIONS

In this section, we give a brief review of stability conditions on derived category of coherent sheaves on a K3 surface, see [Bri07, Bri08] for details.

Suppose X is a complex algebraic K3 surfaces and $\mathcal{D}(X)$ is the bounded derived category of coherent sheaves on X . The Mukai vector for $E \in \mathcal{D}(X)$ is defined as

$$v(E) := (\text{rk}(E), c_1(E), \text{ch}_2(E) + \text{rk}(E)) = \text{ch}(E)\sqrt{\text{td}(X)} \in \mathcal{N}(X)$$

where $\text{ch}(E)$ is the Chern character of E and $\mathcal{N}(X) = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}$ is the numerical Grothendieck group. Recall that the Mukai pairing for $E, E' \in \mathcal{D}(X)$ is given by

$$\langle v(E), v(E') \rangle = c_1(E).c_1(E') - \text{rk}(E).(\text{ch}_2(E') + \text{rk}(E')) - \text{rk}(E').(\text{ch}_2(E) + \text{rk}(E)).$$

The Riemann-Roch theorem shows that for two objects $E, E' \in \mathcal{D}(X)$,

$$\langle v(E), v(E') \rangle = -\chi(E, E') = -\sum_i (-1)^i \dim \text{Ext}^i(E, E').$$

A numerical stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}(X)$ consists of a group homomorphism (central charge)

$$Z: \mathcal{N}(X) \rightarrow \mathbb{C}, \quad Z(E) = \langle \Omega, v(E) \rangle,$$

and a collection of abelian subcategories (semistable objects of phase ϕ) $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ which together satisfy some axioms.

A stability function on an abelian category \mathcal{A} is a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ such that for any non zero object $E \in \mathcal{A}$,

$$Z(E) \in \mathbb{R}^{>0} \exp(i\pi\phi(E)) \text{ with } 0 < \phi(E) \leq 1.$$

Proposition 2.1. [Bri08, Proposition 3.5] *To give a stability condition on a triangulated category \mathcal{D} is equivalent to giving a bounded t-structure on \mathcal{D} and a stability function on its heart which has the Harder-Narasimhan property.*

For a pair $(\beta, \omega) \in \text{NS}(X)$ when ω is an ample divisor, one defines group homomorphism $Z_{(\beta, \omega)}: \mathcal{N}(X) \rightarrow \mathbb{C}$ as

$$Z_{(\beta, \omega)}(E) = \langle \exp(\beta + i\omega), v(E) \rangle$$

and the slope function as

$$\mu_\omega(E) = \frac{c_1(E) \cdot \omega}{\text{rk}(E)}.$$

Consider torsion pair $(\mathcal{T}, \mathcal{F})$ on the category of $\text{Coh}(X)$, where \mathcal{T} consists of sheaves whose torsion free parts have μ_ω -semistable Harder-Narasimhan factors of slope $\mu_\omega > \beta \cdot \omega$ and \mathcal{F} consists of torsion-free sheaves whose μ_ω -semistable Harder-Narasimhan factors have slope $\mu_\omega \leq \beta \cdot \omega$. Tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ gives a bounded t-structure on $\mathcal{D}(X)$ with the heart

$$\mathcal{A}(\beta, \omega) = \{E \in \mathcal{D}(X) : H^i(E) = 0 \text{ for } i \notin \{0, 1\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}.$$

For any choice of (β, ω) , above construction will not give a stability condition on $\mathcal{D}(X)$. Let

$$W(X) = \{(\beta, \omega) : \beta, \omega \in \text{NS} \otimes \mathbb{R}, \omega \text{ is an ample divisor}\} \text{ and}$$

$V(X) = \{(\beta, \omega) \in W(X) : \text{for every } \delta \in \Delta(X) \text{ with } \text{rk}(\delta) > 0, \langle \exp(\beta + i\omega), \delta \rangle \notin \mathbb{R}_{\leq 0}\}$, where $\Delta(X) = \{\delta \in \mathcal{N}(X) : \langle \delta, \delta \rangle = -2\}$ is the root system. Then we have following result.

Proposition 2.2. [Bri08, Lemma 6.2] *For any pair $(\beta, \omega) \in V(X)$, the function $Z_{(\beta, \omega)}$ is a stability function on $\mathcal{A}(\beta, \omega)$ which has the Harder-Narasimhan property. Therefore $\sigma_{(\beta, \omega)} = (Z_{(\beta, \omega)}, \mathcal{A}(\beta, \omega))$ is a Bridgeland stability condition on $\mathcal{D}(X)$.*

Two-dimensional subspace of stability conditions. Let H be a fixed primitive ample divisor on an algebraic K3 surface X . Consider following projection maps:

$$P_1 : \mathcal{N}(X) \rightarrow \mathbb{R}^3, \quad P_1(r, C, s) = \left(r, \frac{C \cdot H}{H^2}, s\right),$$

$$P_2 : \mathbb{R}^3 \setminus \{s = 0\} \rightarrow \mathbb{R}^2, \quad P_2(r, c, s) = \left(\frac{c}{s}, \frac{r}{s}\right),$$

where $d' := H^2/2$. Also $Pr = P_2 \circ P_1$ is their composition.

In this paper, we only focus on a two dimensional subspace $\text{Stab}_H(X)$ which consists of numerical Bridgeland stability conditions $\sigma = (Z_\sigma, \mathcal{A}_\sigma)$ such that skyscraper sheaves at every point are σ -stable of phase one, and the central charge Z_σ factors via P_1 .

Thus, every stability condition $\sigma \in \text{Stab}_H(X)$ is of the form $\sigma_{(bH, wH)} = (Z_{(bH, wH)}, \mathcal{A}(bH, wH))$

for some (b, w) in the upper half plane \mathbb{H} . For simplicity, we denote such a stability condition by $\sigma_{(b,w)} = (Z_{(b,w)}, \mathcal{A}(b, w))$.

Consider the isomorphism

$$k : \text{Stab}_H(X) \rightarrow U_H(X) \quad , \quad \text{where}$$

$$k(\sigma_{(b,w)}) = \text{Pr}(\text{Ker}(Z_{(b,w)})) = \left(\frac{b}{d'(b^2 + w^2)}, \frac{1}{d'(b^2 + w^2)} \right) =: k(b, w),$$

and $U_H(X) = \{k(b, w) : (bH, wH) \in V(X)\} \subset U = \{(x, y) \in \mathbb{R}^2 : y > d'x^2\}$ with the standard topology on \mathbb{R}^2 .

Therefore we can work with the space $U_H(X)$ instead of $\text{Stab}_H(X)$ and apply all known results about the space of stability conditions to this space. By abuse of notations, in the figures we always denote by $\sigma_{(b,w)}$ the corresponding point $k(b, w)$ in $U_H(X)$. For instance, if w is large enough, $\sigma_{(0,w)}$ is a stability condition which is on the y -axis ($k(0, w) = (0, 1/d'w^2)$). Similarly, for any fixed value of b_0 , the stability conditions $\sigma_{(b_0,w)}$ are on the line $y = x/b_0$. When w gets bigger, the corresponding point in $U_H(X)$ gets nearer to the point $(0, 0)$, see Figure 1.

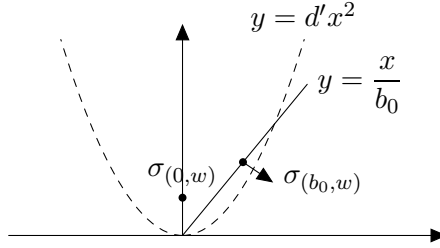


FIGURE 1. $\text{Stab}_H(X) \cong U_H(X)$

Lemma 2.3. *Given a root $\delta = (r, C, s) \in \Delta(X)$. Let I_δ be the set*

$$\{k(b, w) : (b, w) \in \mathbb{H}, Z_{(b,w)}(\delta) \in \mathbb{R}_{\leq 0}\} \subseteq \mathbb{R}^2.$$

Assume pt_δ is the intersection point of the parabola $y = d'x^2$ with the line through origin and $\text{Pr}(\delta)$. Then I_δ is the line segment between $\text{Pr}(\delta)$ and pt_δ .

Proof. For any $(b, w) \in k^{-1}(I_\delta)$,

$$\text{Im}(Z_{(b,w)}(r, C, s)) = wC.H - 2d'bwr = 0 \Rightarrow b = \frac{C.H}{2d'r}.$$

Thus, the point $k(b, w)$ is on the line with equation $y = \frac{x}{b} = \frac{2d'r}{C.H}x$ which passes the point $\text{Pr}(\delta) = (C.H/(2d's), r/s)$. In addition,

$$\text{Re}(Z_{(b,w)}(r, C, s)) = bC.H - s - rd'(b^2 - w^2) \leq 0 \Rightarrow d'w^2 \leq s/r - (C.H)^2/(4r^2d'),$$

and we have equality if $k(b, w) = \text{Pr}(\delta)$, which makes the claim clear. \square

Therefore, $U_H(X)$ is the open subset U minus the line segments I_δ that pass roots (see Figure 2).

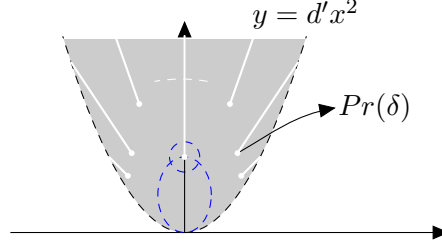


FIGURE 2. $\text{Stab}_H(X)$

Lemma 2.4. *The interior of the ellipse with equation $d'x^2 = y - y^2$ and a sufficiently small punctured disk around the point $(0, 1)$ do not contain any projection $Pr(\delta)$ of a root $\delta \in \Delta(X)$.*

Proof. Let $\delta = (r, C, s)$ be a root, i.e. $\delta^2 = -2$. If $r = 0$, then clearly $Pr(\delta) = (C.H/2d's, 0)$ is outside of the required area. Therefore, we assume $r \neq 0$. By the Hodge index theorem,

$$d' \left(\frac{C.H}{2d's} \right)^2 + \frac{r^2}{s^2} - \frac{r}{s} \geq \frac{C^2}{2s^2} - \frac{r}{s} - \frac{r^2}{s^2} = \frac{r^2 - 1}{s^2} \geq 0$$

which shows $Pr(\delta)$ is not inside the ellipse.

For the second part of lemma, since we only care about an open neighbourhood around $(0, 1)$, we can assume that $r/s < 3/2$ and $r > 1$. Thus

$$\frac{1}{4d'} < \frac{s}{d'r} - \frac{1}{d'r^2} \leq \left(\frac{C.H}{2d'r} \right)^2$$

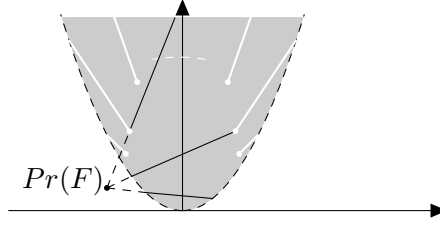
and the claim follows. \square

Remark 2.5. It follows from [Bri08, Proposition 9.3] that for any object $F \in \mathcal{D}(X)$, the space $\text{Stab}_H(X)$ and therefore $U_H(X)$ admit a well-behaved wall and chamber structure controlling stability of F . There exists a locally finite set of walls \mathcal{W}_F of dimension one with following properties:

- (a) When σ varies within a chamber, stability or instability of F does not change.
- (b) When σ lies on a single wall $\mathcal{W}_F \subseteq U_H(X)$, then F is σ -semistable, and if F is stable in one of the adjacent chamber, then it is unstable in the other adjacent chamber.

The next lemma describes walls \mathcal{W}_F in $U_H(X)$.

Lemma 2.6. *Let $\sigma_{(b,w)} \in \text{Stab}_H(X)$ be a stability condition and $E, E' \in \mathcal{A}(b, w)$ are two $\sigma_{(b,w)}$ -semistable objects. Then, E and E' have the same phase if and only if the points $k(b, w)$, $Pr(v(E))$ and $Pr(v(E'))$ are collinear. In particular, the walls of stability \mathcal{W}_F for any $F \in \mathcal{D}(X)$, are segments of the lines passing through $Pr(v(F))$, see Figure 3.*

FIGURE 3. Walls \mathcal{W}_F

Proof. The two semistable objects $E, E' \in \mathcal{A}(b, w) = \mathcal{P}(0, 1]$ have the same phase if and only if $Z_{(b, w)}(E) = \alpha Z_{(b, w)}(E')$ for some $\alpha \in \mathbb{R}^{>0}$, which means

$$\alpha P_1(v(E)) - P_1(v(E')) \in P_1(\text{Ker}(Z_{(b, w)})).$$

Or, equivalently, the points $P_2(1, b, d'(b^2 + w^2)) = k(b, w)$, $Pr(v(E))$ and $Pr(v(E'))$ are collinear. Indeed, $P_1(\text{Ker}(Z_{(b, w)})) \subset P_1(\mathcal{N}(X)) \otimes \mathbb{R}$ is a one dimensional subspace that can be generated by the vector $(1, b, d'(b^2 + w^2))$.

Therefore, the set of points $k(b, w)$ that any fixed object E can be $\sigma_{(b, w)}$ -stable factor of F , is precisely segment of the line that connects $Pr(F)$ to $Pr(E)$, see [Bri08] for details. \square

Relation to slope-stability. Definition of stability conditions $\sigma_{(\beta, \omega)}$ for some $(\beta, \omega) \in V(X)$ are based on slope-stability of torsion free sheaves. The following well-known Lemma makes clear the relation between these two notions of stability.

Lemma 2.7. *Let E be a locally-free sheaf of positive rank and $\sigma_{(\beta, \omega)} = (Z_{(\beta, \omega)}, \mathcal{A}(\beta, \omega))$ be a stability condition on $\mathcal{D}(X)$. Then E is μ_ω -stable with slope $\beta.\omega$ if and only if $E[1]$ is $\sigma_{(\beta, \omega)}$ -stable of phase 1.*

Proof. Assume $E[1] \in \mathcal{A}(\beta, \omega)$ is $\sigma_{(\beta, \omega)}$ -stable of phase one.

$$Z_{(\beta, \omega)}(E[1]) = \langle v(E[1]), e^{\beta + i\omega} \rangle = \left\langle (r, C, s), (1, \beta, \frac{\beta^2 - \omega^2}{2}) \right\rangle + i \langle (r, C, s), (0, \omega, \beta.\omega) \rangle.$$

Since the imaginary part vanishes,

$$\text{Im}(Z(E[1])) = C.\omega - r\beta.\omega = 0 \Rightarrow \mu_\omega(E) = \frac{C.\omega}{r} = \beta.\omega.$$

Moreover, by definition of the heart, $E \in \mathcal{F}_{(\beta, \omega)}$ and all μ_ω -semistable HN factors of E have slope less than or equal to $\beta.\omega$. Therefore E is μ_ω -semistable.

Now assume for a contradiction there exists a proper torsion free quotient sheaf ($F' = E/F$) in $\text{Coh}(X)$ with the same μ_ω -slope. So, we have exact sequence $0 \rightarrow F \rightarrow E \rightarrow F' \rightarrow 0$ in $\text{Coh}(X)$ where all three sheaves E , F and F' are μ_ω -semistable torsion free sheaf of slope $\beta.\omega$. By definition, all these three sheaves are in $\mathcal{F}_{(\beta, \omega)}$ and we have following exact sequence in the abelian category $\mathcal{P}[1]$.

$$0 \rightarrow F[1] \rightarrow E[1] \rightarrow F'[1] \rightarrow 0.$$

which is a contradiction.

For the converse, assume E is a μ_ω -slope-stable locally-free sheaf of slope $\beta.\omega$, so $E \in \mathcal{F}_{(\beta,\omega)}$ and $\text{Im}(Z_{(\beta,\omega)}(E)) = 0$. Therefore $E \in \mathcal{P}[1]$ and E is $\sigma_{(\beta,\omega)}$ -semistable object. Assume for a contradiction that E is strictly $\sigma_{(\beta,\omega)}$ -semistable. [Bri08, Lemma 10.1] implies that the every stable object of phase one is a skyscraper sheaf or shift $F[1]$ of a locally-free sheaf. Since E is a locally-free sheaf,

$$\text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_x, E[1]) = \text{Hom}_{\mathcal{D}(X)}(E[1], \mathcal{O}_x) = 0$$

Therefore, all stable factors of $E[1]$ are shift of locally-free sheaves which implies that E has a subsheaf in $\text{Coh}(X)$ with the same slope and smaller rank, a contradiction. \square

3. MERCAT'S CONJECTURE

In this section, we always assume (X, H) is a smooth polarized K3 surface over \mathbb{C} which satisfies condition $(*)$. Like before, we use the notation $d' := H^2/2$. Let F be a locally free sheaf which is μ_H -stable and has Mukai vector $v(F) = (r, C, s)$. Therefore, Lemma 2.7 gives $\sigma_1 = \sigma_{(b_1, w_1)}$ -stability of F , where

$$b_1 = \frac{C.H}{2d'r} \quad \text{and} \quad d'w_1^2 > 1.$$

Hence, the corresponding point $k(b_1, w_1)$ is on the line segment that connects $Pr(F)$ to origin, see Figure 4. Similarly, the shift of twisted sheaf $F(-H)$ with the Mukai vector

$$v(F(-H)[1]) = (-r, -C + rH, -s - d'r + C.H),$$

is $\sigma_2 = \sigma_{(b_2, w_2)}$ -stable where

$$b_2 = \frac{H.(C - rH)}{2d'r} = b_1 - 1 \quad \text{and} \quad d'w_2^2 > 1.$$

Lemma 3.1. *Given real number b where $b_1 < b < b_2$, if*

$$(7) \quad d'w_b^2 = -d'b^2 + b(-d' + \frac{C.H}{r}) + \frac{C.H}{2r} - \frac{1}{2d'}\left(\frac{C.H}{r}\right)^2 + \frac{s}{r} > 1,$$

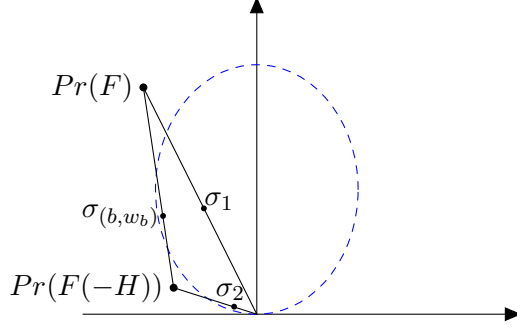
the pair (b, w_b) gives a stability condition $\sigma_{(b, w_b)}$ such that F and $F(-H)[1]$ have the same phase.

Proof. To find the expression for w_b , it is enough to check the intersection point of the line through $Pr(F)$ and $Pr(F(-H))$ with the line $y = x/b$. If $d'w_b^2 > 1$, Lemma 2.4 implies that we have stability condition $\sigma_{(b, w_b)}$. In addition, since $b_2 < b < b_1$, the objects F and $F(-H)[1]$ are in the heart $\mathcal{A}(b, w_b)$ and Lemma 2.6 implies that F and $F(-H)[1]$ have the same phase in the stability condition $\sigma_{(b, w_b)}$. \square

Equation (7) shows that for the critical point $b_3 = b_1 - 1/2$, we have the maximum value

$$d'w_3^2 = \frac{d'}{4} - \frac{\Delta_H(F)}{2r^2},$$

which is greater than one if the sheaf F satisfies inequality (3). Hence, we have stability condition $\sigma_3 = \sigma_{(b_3, w_3)}$ such that F and $F(-H)[1]$ have the same phase. The next step

FIGURE 4. Stability conditions on the line segment P_1P_2

towards proof of Theorem 1.1 is to show F and $F(-H)[1]$ are both σ_3 -stable. Therefore we must check the walls of stability of F and $F(-H)[1]$.

Lemma 3.2. *Let F be an object in $\mathcal{D}(X)$ and $\sigma \in \text{Stab}_H(X)$ be a stability condition. Let $\Lambda' \subset \Lambda = P_1(\mathcal{N}(X))$ be a sublattice such that the quotient Λ/Λ' can be generated by $v(F)$. Then F cannot be strictly σ -semistable if $P_1(\text{Ker}(Z_\sigma)) \subset \Lambda' \otimes \mathbb{R}$.*

Proof. Assume for a contradiction that F is strictly σ -semistable and E is one of its σ -stable factors. By the assumptions, there are $v' \in P_1(\text{Ker}(Z_\sigma))$ and $v'' \in \Lambda'$ such that any element of the lattice Λ , and so $P_1(v(E))$, can be written as

$$P_1(v(E)) = xP_1(v(F)) + yv' + zv'',$$

for some $x \in \mathbb{Z}$ and $y, z \in \mathbb{R}$. The objects F and E have the same phase. Hence $z = 0$ and $Z_\sigma(v(E)) = xZ_\sigma(v(F))$, which is in contradiction to $0 < |Z_\sigma(v(E))| < |Z_\sigma(v(F))|$. \square

Proof of Theorem 1.1. By the assumption, the equation

$$(8) \quad mr - n\left(\frac{C.H}{2d'}\right) = -1 \quad m, n \in \mathbb{N}$$

has always a solution (m_0, n_0) with $0 < n_0 < r$.

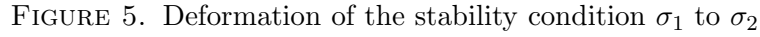
Now, consider a straight line path in $\text{Stab}_H(X)$ that starts at the stability condition σ_1 and go to the stability condition σ_3 . If this path hits any wall \mathcal{W}_F , then as it is shown in Figure 5, that wall would also intersect the line segment $y = x/b_4$ for

$$b_4 = \frac{C.H}{2d'r} - \frac{1}{rn_0} = \frac{m_0}{n_0}.$$

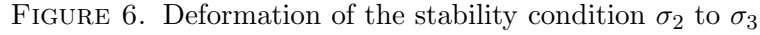
Inequality (3) implies that the intersection point is always of form $k(b_4, w_4)$ when

$$d'w_4^2 \geq \frac{d'}{r(r-1)} - \frac{d'}{r^2(r-1)^2} - \frac{\Delta_H(F)}{2r^2} > 1.$$

Thus, the intersection point is always a stability condition $\sigma_4 = \sigma_{(b_4, w_4)}$. Consider the sublattice $\Lambda' \subset P_1(\mathcal{N}(X))$ which is generated by $(0, 0, 1)$ and $(n_0, m_0, 0)$. Then, clearly $P_1(\text{Ker}(Z_\sigma)) \in \Lambda' \otimes \mathbb{R}$ and $P_1(v(F))$ can generate the quotient $P_1(\mathcal{N}(X))/\Lambda'$. Hence,



Similarly, for proving σ_3 -stability of $F(-H)$, it is enough to consider the path which starts at σ_2 go straight to σ_3 . As it is shown in Figure 6, if the path hits any wall $\mathcal{W}_{F(-H)}$, then


$$b_5 = \frac{m_1}{n_1} - 1 = \frac{C.H}{2d'r} - 1 - \frac{1}{rn_1},$$
$$d'w_5^2 \geq \frac{d'}{r(r-1)} - \frac{d'}{r^2(r-1)^2} - \frac{\Delta_H(F)}{2r^2} > 1.$$

Now, consider the following distinguished triangle in $\mathcal{D}(X)$ for $C \in |H|$,

$$F \rightarrow F|_C \rightarrow F(-H)[1].$$

The objects F and $F(-H)[1]$ are σ_3 -stable of the same phase. Therefore, $F|_C$ is σ_3 -semistable with F and $F(-H)[1]$ as its σ_3 -stable factors. Moreover,

$$\operatorname{Rel}[Z_{(b_3, w_3)}(F(-H)[1])] = \operatorname{Rel}[Z_{(b_3, w_3)}(F)] = \operatorname{Rel}[Z_{(b_3, w_3)}(F|_C)] = 0$$

and if $w > w_3$,

$$\operatorname{Rel}[Z_{(b_3, w)}(F(-H)[1])] < 0, \quad \operatorname{Rel}[Z_{(b_3, w)}(F)] > 0,$$

and

$$\operatorname{Rel}[Z_{(b_3, w)}(F|_C)] = 0,$$

which shows $\sigma_{(b_3, w)}$ -strict stability of $F|_C$ where $w_3 < w < w_3 + \epsilon$ and ϵ is a sufficiently small positive number.

Now assume \tilde{F} is a subsheaf of $F|_C$. By definition, the torsion sheaves \tilde{F} and $(F|_C)/\tilde{F}$ on the surface X are in the heart $\mathcal{A}(b_3, w)$. Assume the Mukai vector of \tilde{F} as a sheaf on X is $v(\tilde{F}) = (0, \tilde{C}, \tilde{s})$. Then $\sigma_{(b_3, w)}$ -stability of $F|_C$ gives

$$\operatorname{Rel}[Z_{(b_3, w)}(\tilde{F})] = b_3 \tilde{C} \cdot H - \tilde{s} > 0 \quad \Rightarrow \quad \mu(\tilde{F}) < \mu(F|_C),$$

because

$$\mu(\tilde{F}) = \frac{\deg_{C'}(\tilde{F})}{\operatorname{rk}(\tilde{F})} = (g-1) \left(\frac{2\tilde{s}}{\tilde{C} \cdot H} + 1 \right).$$

Therefore, $F|_C$ is a slope-stable vector bundle on the curve C . \square

Let A be a globally generated line bundle on a smooth curve $C \in |H|$. We fix the notations $g = g(C)$, $h := h^0(C, A) - 1$, $d := \deg_C(A)$ and $h' := h^1(C, A)$. The Riemann-Roch theorem gives

$$h + 1 - h' = -d' + d = 1 - g + d.$$

The vector bundle $F = E_{C,A}^\vee$ which has been defined via exact sequence (2), has Mukai vector

$$v(F) = (h + 1, -H, h')$$

and is μ_H -stable (see [BF15]). Therefore, Corollary 1.2 is the direct result of Theorem 1.1. The main theorem of [Laz86] implies that for any smooth curve $C \in |H|$ with genus g and given integers r and d , there exists a globally generated line bundle A on the curve C with $h^0(C, A) = h + 1$ and $\deg_C(A) = d$ if and only if

$$\rho(h, d, g) = g - (h + 1)(g - d + h) \geq 0.$$

As a result, we have

$$\operatorname{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor.$$

Proof of Corollary 1.3. First of all, existence of a line bundle on the smooth curve $C \in |H|$ with $h^0(C, A) = h + 1$ and $\deg_C(A) = d$ implies

$$(9) \quad \rho(h, d, d) \geq 0.$$

Dualizing the exact sequence (2) gives

$$0 \rightarrow H^0(A)^\vee \otimes \mathcal{O}_X \rightarrow E_{C,A} \rightarrow \omega_C \otimes A^\vee \rightarrow 0$$

which is exact on global sections. Thus

$$h^0(C', E_{C,A}|_{C'}) = h^0(X, E_{C,A}) = h^0(C, A) + h^1(C, A) = 2(h+1) + d' - d.$$

In addition, $\deg_{C'}(E_{C,A}|_{C'}) = C' \cdot c_1(E_{C,A}) = 2d'$ and $r(E_{C,A}|_{C'}) = h+1$, see [Laz86, MAO14] for details.

If the bundle $E_{C,A}|_{C'}$ contributes to $\text{Cliff}_n(C')$, it must satisfy the two conditions in the definition of $\text{Cliff}_n(C')$,

$$(10) \quad \deg_{C'}(E_{C,A}|_{C'}) \leq r(E_{C,A}|_{C'}) \cdot (g-1) \Rightarrow 2 \leq (h+1),$$

$$(11) \quad h^0(C', E_{C,A}|_{C'}) \geq 2 \cdot r(E_{C,A}|_{C'}) \Rightarrow (h+1) \leq h'.$$

Finally, the bundle $E_{C,A}|_{C'}$ invalidates the Mercat conjecture if

$$(12) \quad \text{Cliff}(E_{C,A}|_{C'}) < \text{Cliff}(C') = \lfloor \frac{g-1}{2} \rfloor.$$

The stated inequalities in the corollary are result of (9), (10), (11) and (12). \square

Remark 3.3. One can use the same method as in [Bay16b] to describe the loci of μ_H -stable sheaves on a K3 surface with fixed Mukai vector (r, H, s) and fixed number of global sections. However, their restrictions to smooth curves do not give additional examples of triples (r, d, g) for which there exists a counterexample to Mercat's conjecture, other than those described in Corollary 1.3.

4. SLOPE-STABILITY OF M_L

In this section, we provide a new proof for Camere's result on the stability of M_L . Assume X is a complex algebraic K3 surface which not necessarily satisfies condition (*), and L is an ample line bundle which is generated by global sections. Also assume $L = lH$ where H is a primitive ample divisor. The corresponding vector bundle M_L is the kernel of evaluation map on global sections of L as defined in (6).

We will write $d' := H^2/2$ and $h := h^0(X, L) - 1 = l^2 d' + 1$. In the Figure 7, we denote the projection points by $M = \text{Pr}(v(M_L)) = (-l, h)$, $L = \text{Pr}(v(L)) = (l/h, 1/h)$, $O' = \text{Pr}(v(\mathcal{O}_X)) = (0, 1)$ and O for the origin. Let \mathcal{S} be the set of all points inside or on the boundary of triangle $\triangle MLO$ but not the points M , O' , L and O .

Lemma 4.1. *The set \mathcal{S} does not contain any projection $\text{Pr}(\delta)$ of a root $\delta = (r, C, s) \in \Delta(X)$.*

Proof. The claim is clear for the triangle $O'LO$, because the points L and O' are on the ellipse with equation $y - y^2 - d'x^2 - 1 = 0$ and Lemma 2.4 implies that there is no $\text{Pr}(\delta)$ inside the ellipse.

Similar to the argument of Lemma 2.4, if $r \geq (h+1)$, the point $\text{Pr}(\delta)$ cannot be inside of ellipse with equation $(h+1)^2(y^2 - d'x^2) - y = 0$. The point M is on this ellipse, thus there is no $\text{Pr}(\delta)$ with $|r| > (h+1)$ in \mathcal{S} .

Assume for a contradiction, there is a point $Pr(\delta)$ with $|r| \leq (h+1)$ inside or on the boundary of triangle MOO' . So we have $r/s > 0$ and by Hodge index theorem

$$\frac{l^2}{h^2} \leq \frac{2h|s| - 2}{2d'h^2} < \frac{2rs - 2}{2d'r^2} \leq \left(\frac{C.H}{2d'r} \right)^2,$$

which shows that the point $Pr(\delta)$ cannot be above the line $x = -\frac{l}{h}y$, a contradiction. \square

Recall that an object $S \in \mathcal{D}(X)$ is called spherical if $\text{Hom}_{\mathcal{D}(X)}(S, S[i]) = \mathbb{C}$ for $i = 0, 2$ and it is zero otherwise. For any stability condition σ on $\mathcal{D}(X)$, [BB13, Lemma 2.5] implies that all stable factors of a spherical object are also spherical. The vector bundles L , \mathcal{O}_X and M_L are spherical objects. Hence, their stable factors for any stability condition must be also spherical. Now, we can use the same argument as [Bay16a, Lemma 4.2] to check the walls of stability of these objects.

Lemma 4.2. *Let $F \in D(X)$ be a spherical object with the Mukai vector $v(F) = (r', C', s')$ which has negative discriminant,*

$$\Delta_H(F) = \frac{(C'.H)^2}{H^2} - 2r's' < 0.$$

Then, there is no wall \mathcal{W}_F ending at the point $Pr(F)$.

Proof. Assume for a contradiction that there is such a wall. Hence, there is a fixed spherical object $E \in D(X)$ with $v(E) \neq v(F)$ such that

$$(13) \quad 0 < |Z_{(b,w)}(v(E))| < |Z_{(b,w)}(v(F))|,$$

for all stability conditions $\sigma_{(b,w)}$ which are on the wall \mathcal{W}_F and also sufficiently close to $Pr(v(F))$. Since $Pr(v(F)) \neq Pr(v(E))$, there exists $k > 0$ such that $k < |Z_{(b,w)}(v(E))|$ for all such stability conditions. However, by definition $|Z_{(b,w)}(v(F))|$ goes to zero when $k(b, w)$ gets near to the point $Pr(F)$, which is in contradiction to (13). \square

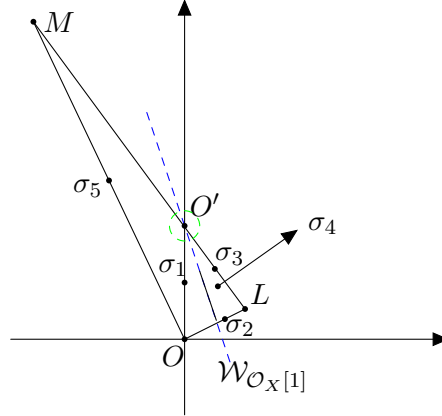
In particular, we have following result.

Lemma 4.3. *There are no walls $\mathcal{W}_{\mathcal{O}_X[1]}$, \mathcal{W}_L and $\mathcal{W}_{M_L[1]}$ in the interior of \mathcal{S} .*

Proof. Assume for a contradiction that there exists such a wall $\mathcal{W}_{\mathcal{O}_X[1]}$. Lemma 2.6 implies that $\mathcal{W}_{\mathcal{O}_X[1]}$ is segment of a line passing through $Pr(\mathcal{O}_X[1])$. Also, lemma 4.1 shows that each point in the area \mathcal{S} is in correspondence with a unique geometric stability condition. Hence, as it is shown in Figure 7 one of the endpoints of the wall must be $Pr(\mathcal{O}_X[1])$, which is in contradiction to Lemma 4.2. A similar argument also shows that there is no wall \mathcal{W}_L and $\mathcal{W}_{M_L[1]}$ in the interior of \mathcal{S} . \square

By the same argument as Lemma 4.3, there are also no walls for L and $\mathcal{O}_X[1]$ on the line segment $\overline{O'L}$. Similarly, the wall $\mathcal{W}_{M_L[1]}$ cannot be on the line segment \overline{MO} .

Lemma 4.4. *Let $\sigma_3 = \sigma_{(b_3, w_3)}$ be a stability condition with $k(b_3, w_3)$ on the line segment $\overline{LO'}$. If L and $\mathcal{O}_X[1]$ are σ_3 -stable, then $M_L[1]$ is $\sigma_4 = \sigma_{(b_3, w_4)}$ -strictly stable for $w_4 > w_3$.*

FIGURE 7. Slope stability of M_L .

Proof. Since L and $\mathcal{O}_X[1]$ are σ_3 -stable and $\text{Hom}(\mathcal{O}_X, M_L) = 0$, dual of [BM11, Lemma 5.9] implies that σ_3 is on the closure of stability area of $M_L[1]$, i.e., if we change stability condition σ_3 in the right direction, we will reach its strict stability. To find the correct direction, it is enough to check the phase of its subobject L .

By definition, the heart $\mathcal{A}(b_3, w)$ is an abelian category which does not depend on w , and the ordering of phase function is the same as ordering of function $-\frac{\text{Rel}(Z(E))}{\text{Im}(Z(E))}$. Hence, there are two functions with respect to w :

$$\begin{aligned}\varphi_1(w) &:= \frac{-(2d'w)\text{Rel}(Z_{(b_3,w)}(\mathcal{O}_X[1]))}{\text{Im}(Z_{(b_3,w)}(\mathcal{O}_X[1]))} = \frac{-1 - d'b_3^2 + d'w^2}{b_3}, \\ \varphi_2(w) &:= \frac{-(2d'w)\text{Rel}(Z_{(b_3,w)}(L))}{\text{Im}(Z_{(b_3,w)}(L))} = \frac{-1 - d'(b_3 - l)^2 + d'w^2}{b_3 - l}.\end{aligned}$$

For $w = w_3$, we have equality $\varphi_1(w_3) = \varphi_2(w_3)$. Also, $\varphi_2(w) > \varphi_1(w)$ for $w < w_3$ which means $M_L[1]$ is not $\sigma_{(b_3,w)}$ -stable. Thus, we have $\sigma_{(b_3,w)}$ -stability of $M_L[1]$ for $w > w_3$. \square

Proof of Theorem 1.4. The structure sheaf \mathcal{O}_X is μ_H -stable of slope zero. Hence, Lemma 2.7 gives $\sigma_1 = \sigma_{(0,w_1)}$ -stability of $\mathcal{O}_X[1]$ when $d'w_1^2 > 1$. Also, by definition, $\mathcal{O}_X[1]$ is in the heart $\mathcal{A}(b, w)$ for $b > 0$. Similarly, the line bundle L is μ_H -stable of slope $2d'l$, thus it is $\sigma_2 = \sigma_{(b_2,w_2)}$ -stable for $b_2 = l$ and $d'w_2^2 > 1$ and it is in the heart $\mathcal{A}(b, w)$ for $b < l$.

Let $\sigma_3 = \sigma_{(b_3,w_3)}$ be a stability condition which its corresponding point $k(b_3, w_3)$ is on the line segment \overline{LO} , see Figure 7. Lemma 4.3 shows that there are no wall for L and $\mathcal{O}_X[1]$ in the interior of \mathcal{S} . Therefore, we have σ_3 -stability of both L and $\mathcal{O}_X[1]$. Moreover, Lemma 2.6 implies that they have the same phase in this stability condition. Hence, the distinguished triangle

$$L \rightarrow M_L[1] \rightarrow \mathcal{O}_X^{\oplus(h+1)}[1],$$

gives σ_3 -semistability of $M_L[1]$. Lemma 4.4 shows that if we deform stability condition σ_3 towards the inside of triangle $\triangle MLO$, we can reach strict stability of $M_L[1]$. Denote this new stability condition by σ_4 . Again, since there is no wall $\mathcal{W}_{M_L[1]}$ in the interior of \mathcal{S} , we have $\sigma_5 = \sigma_{(b_5, w_5)}$ -stability of $M_L[1]$ where $k(b_5, w_5)$ is on the line segment \overline{MO} . Thus, Lemma 2.7 gives μ_H -stability of M_L . \square

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